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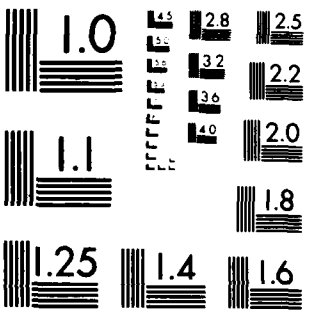

A UNIFORM BOUND FOR THE TAIL PROBABILITY OF KOLMOGOROV-SMIRNOV STATISTICS

by
Inchi Hu
Stanford University

TECHNICAL REPORT NO. 28
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DEPARTMENT OF STATISTICS
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1. Introduction

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with a continuous but unknown distribution function F. Denote the empirical distribution function for sample $X_1, X_2, \cdots, X_n$ by $\hat{F}(x) = \frac{1}{n}\{\#\text{ of } X_i \le x,\ i = 1, \cdots, n\}$. In testing goodness of fit, that is, we want to test $F = F_0$ for some specific choice of F_0, the commonly used test statistics are

$$D_n^+ = \sqrt{n}\sup(\hat{F}_n(x) - F(x))$$

$$D_n^- = \sqrt{n}\inf_x(\hat{F}(x) - F(x))$$

$$D_n = \sqrt{n}\sup_x|\hat{F}(x) - F(x)|.$$

The purpose of this paper is to give a bound for the tail probability of D_n^- in the following form.

Theorem 1. $p\{D_n^- > \sqrt{n}\zeta\} \le 2\sqrt{2}e^{-2n\zeta^2}$.

A bound of the form $p\{D_n^- > \sqrt{n}\zeta\} \le C\ e^{-2n\zeta^2}$, where C is some unspecified constant, has been proven by Dvoretzky, Kiefer, and Wolfowitz (1956). There are several papers conjecturing that C can be taken as 1, cf. Birnbaum and McCarty (1958) and Csörgő and Horváth (1981). Each of them is substantiated by considerable numerical computation, although no proof is available. Devroye and Wise (1979) proved $C \le \left\{2 + 32/(6\pi)^{\frac{1}{2}} + 8/3^{\frac{1}{2}} + 2^{\frac{1}{2}}\ 4\ \exp\left(\frac{71}{16}\right)\right\} \le 306$, but this bound is too large to be useful in any application. The best result known to the author (before this paper was written) is $c \le 29$, due to G. Shorack (private communication), so the result of this paper is a substantial improvement of all the results known so far and partial support of the conjecture

2. Proof of the Main Result

First we introduce some notation and basic facts about exponential families. Assume the distribution function F of X_1 can be imbedded in an exponential family, i.e. for all θ in some neighborhood of 0 $\exp[\psi(\theta)] = \int \exp(\theta x)\ F(dx)$ is finite, so $\exp[\theta x - \psi(\theta)]\ F(dx)$ defines a family of probability distributions indexed by θ.

It is easy to show that the mean and variance of these distributions are given by $\psi'(\theta)$ and $\psi''(\theta)$ respectively. Hence $\mu = \psi'(\theta)$ is a one to one function of θ. It will be convenient to regard this family of distributions as indexed by μ and write $F_\mu(dx) = \exp[\theta x - \psi(\theta)]\ F(dx)$. Let P_μ denote the probability according to which $X_1, X_2 \cdots$ are independent with $P_\mu(X_i \in dx) = F_\mu(dx)$ $(i = 1, 2, \cdots)$. The density of $S_n = X_1 + \cdots + X_n$ under P_μ will be denoted by $f_{\mu,n}$. If A is an event belonging to the σ -field generated by $X_1, \cdots X_m$, the following notation will be used: $P_\zeta^{(m)}(A) = P_\mu(A|S_m = m\zeta)$. In this paper we consider only events of the form $A = \{\tau < k\}$, $(k = 1, 2, \cdots m)$, where τ is a stopping time.

Siegmund (1982) derived the following fundamental identity

$$P_{\mu_0}^{(m)}(\tau < k) = \exp\{-m[(\theta_2 - \theta_0)\ \mu_0 + \psi(\theta_0) - \psi(\theta_2)\} \int_{\{\tau<k\}} f_{\mu_2,\, m-\tau}(m\mu_0 - S_\tau) \exp[-(\theta_1 - \theta_2)\ S_\tau]/f_{\mu_0,\, m}(m\mu_0) dP_{\mu_1} \tag{1}$$

The notation $\mu_i = \mu(\theta_i)$, $i = 0, 1, 2$ is used above, and θ_1, θ_2 satisfy $\psi(\theta_1) = \psi(\theta_2)$.

Let us bring our attention back to D_n^-. It is well known that the distribution of D_n^- is the same for all continuous distributions, so without loss of generality we may take F to be the uniform distribution on $(0, 1)$. The well known representation of uniform order statistics in terms of sums of independent exponential random variables shows that

$$\begin{aligned} P\{D_n^- > \sqrt{n}\,\zeta\} &= P\{\sup_{0<x<1}(x - \hat{F}_n(x) > \zeta\} \\ &= P\{\max_{1\le j\le n}(W_j - j) \ge n\zeta - 1 | W_{n+1} - (n+1) = -1\} \\ &= P_{\mu_0}^{(m)}\{\tau < m\} \end{aligned}$$

where $W_j = Y_1 + \cdots + Y_j$ and $Y_1, Y_2, \cdots$ are independent standard exponential, $m = n + 1$, $\mu_0 = \frac{-1}{m}$, $\tau = \inf\{i :\ W_i - i \ge n\zeta - 1\}$.

For reasons which will be indicated later, we divide the set $\{\tau < m\}$ into two parts $\{\tau \ge \frac{n}{2} + 1\} \cup \{\frac{n}{2} + 1 < \tau < m\}$ and apply a time reversal argument to the later part, i.e.

$$\begin{aligned} P_{\mu_0}^{(m)}(\tau < m) &= P_{\mu_0}^{(m)}\left(\tau \le \frac{n}{2} + 1\right) + P_{\mu_0}^{(m)}\left(\frac{n}{2} + 1 < \tau < m\right) \\ &\le P_{\mu_0}^{(m)}\left(\tau \le \frac{n}{2} + 1\right) + \tilde{P}_{\mu_0}^{(m)}\left(\tau < \frac{n}{2}\right) \end{aligned}$$

where $\nu_0 = \frac{1}{m}$, $T = \inf\{i : S_i \geq n\zeta\}$ and under the probability $\tilde{P}$ S_i has the same distribution as $i - W_i$ $(i = 1, \cdots, n+1)$. By (1) we have

$$P_{\mu_0}^{(m)}\{\tau \leq \frac{n}{2} + 1\} = \exp\{-m[(\theta_2 - \theta_0)\,\mu_0 + \psi(\theta_0) - \psi(\theta_2)]\} \tag{2}$$
$$\cdot \int_{(\tau \leq \frac{n}{2}+1)} f_{\mu_2,\, m-\tau}(m\mu_0 - S_\tau)\, \exp[-(\theta_1 - \theta_2)\, S_\tau] / f_{\mu_0,\, m}(m\mu_0)\, dP_{\mu_1}$$

and

$$\tilde{P}_{\nu_0}^{(m)}\left(T < \frac{n}{2}\right) = \exp\{-m[(\lambda_2 - \lambda_0)\,\nu_0 + \phi(\lambda_0) - \phi(\lambda_2)]\} \tag{3}$$
$$\cdot \int_{(T < \frac{n}{2})} g_{\nu_2,\, m-T}(m\nu_0 - S_T) \exp[-(\lambda_1 - \lambda_2)\, S_T] / g_{\nu_0,\, m}(m\nu_0)\, d\tilde{P}_{\nu_1},$$

where

$$\begin{aligned}
\psi(\theta) &= -\theta - \log(1-\theta), \\
\phi(\lambda) &= \lambda - \log(1+\lambda), \\
\mu(\theta) &= \psi'(\theta) = \frac{\theta}{1-\theta}, \\
\nu(\lambda) &= \psi'(\lambda) = \frac{\lambda}{1+\lambda},
\end{aligned}$$

$$f_{\mu,k}(x) = \frac{(1-\theta)^k}{(k-1)!}(x+k)^{k-1}\exp[(-x+k)(1-\theta)],\ x \geq -k,\ -\infty < \theta < 1,$$
$$g_{\nu,k}(y) = \frac{(1+\lambda)^k}{(k-1)!}(k-y)^{k-1}\exp[(1+\lambda)(y-k)],\ y \leq k,\ -1 < \lambda < \infty,$$

$\theta_2 < 0 < \theta_1 < 1$ satisfy $\psi(\theta_2) = \psi(\theta_1)$, and $-1 < \lambda_2 < 0 < \lambda_1$ satisfy $\phi(\lambda_2) = \phi(\lambda_1)$.

We work with (2) first. Under P_{μ_1} the increment of the random walk S_i has an exponential right tail. The following Lemma is a direct consequence. The proof is omitted.

Lemma 1. *Under P_{μ_1} $R_m = S_\tau - (n\zeta - 1)$ is independent of τ and has an exponential distribution with parameter $(1 - \theta_1)$.*

By Lemma 1

$$\begin{aligned}
\int_{(r\le \frac{n}{2}+1)} f_{\mu_2,m-r}(m\mu_0 &- S_r)\exp[-(\theta_1-\theta_2)\,S_r]dP_{\mu_1}/f_{\mu_0,m}(m\mu_0)\\
&= \sum_{k=1}^{[\frac{n}{2}+1]}\int_{(r=k)} f_{\mu_2,m-k}(-n\zeta - R_m)\exp[-(\theta_1-\theta_2)R_m]dP_{\mu_1}\\
&\cdot\ \exp[-(\theta_1-\theta_2)(n\zeta-1)]/f_{\mu_0,m}(m\mu_0)\\
&= (1-\theta_1)\exp[-(\theta_1-\theta_2)(n\zeta-1)]\sum_{k=1}^{[\frac{n}{2}+1]} P_{\mu_1}(\tau=k)\\
&\cdot\ \int_0^{m-k-n\zeta} f_{\mu_2,m-k}(-n\zeta-x)\exp[-x(1-\theta_2)]dx/f_{\mu_0,m}(m\mu_0)\\
&= \exp[-(\theta_1-\theta_2)n\zeta]\sum_{k=1}^{[\frac{n}{2}+1]} P_{\mu_1}(\tau=k)\, f_{\mu_2,m-k+1}(-n\zeta-1)/f_{\mu_0,m}(\mu_0 m).
\end{aligned}$$

Observe that $f_{\mu_2,m-k+1}(x)$ is maximized at $x=\frac{(m-k+1)\theta_2-1}{1-\theta_2}$ and the maximized value is

$$\frac{(1-\theta_2)(m-k)^{m-k}\,e^{-(m-k)}}{(m-k)!},$$

and

$$f_{\mu_0,m}(m\mu_0)=\frac{m^m\,e^{-m}}{(m-1)[(m-1)!]}.$$

Substituting these results into the expression above, we have an upper bound of the form

$$(1-\theta_2)\exp[-(\theta_1-\theta_2)n\zeta]\sum_{k=1}^{[\frac{n}{2}+1]} P_{\mu_1}(\tau=k)\frac{(m-k)^{m-k}\,e^{-(m-k)}\,(m-1)[(m-1)!]}{(m-k)!\,m^m\,e^{-m}}.$$

Using Stirling's formula with upper and lower bound (see e.g. Feller, Vol. I, page 54), we find the expression above is bounded by

$$\begin{aligned}
(1 &- \theta_2)\exp[-(\theta_1-\theta_2)n\zeta]e\left(\frac{m-1}{m}\right)^m\sum_{k=1}^{[\frac{n}{2}+1]} P_{\mu_1}(\tau=k)\left(\frac{m-1}{m-k}\right)^{\frac{1}{2}}\\
&\le (1-\theta_2)\exp[-(\theta_1-\theta_2)n\zeta]\,e\left(\frac{m-1}{m}\right)^m\sqrt{2}.
\end{aligned}$$

So $P_{\mu_0}^{(m)}\left\{\tau\le\frac{n}{2}+1\right\}\le\sqrt{2}\,\exp\{-n[(\theta_1-\theta_2)\zeta-\psi(\theta_2)]\}$.

The process for bounding (3) is more or less the same, although we lose the independence of $S_T - n\zeta$ and T.

$$\begin{aligned}
&\int_{(T<\frac{n}{2})} g_{\nu_2,m-T}(m\nu_0 - S_T)\exp[-(\lambda_1-\lambda_2)S_T]d\tilde{P}_{\nu_1}/g_{\nu_0,m}(\nu_0 m) \\
&\quad\le \sum_{k=1}^{[\frac{n}{2}]}\int_{n\zeta}^{n\zeta+1} g_{\nu_2,m-T}(1-y)\ \exp[-(\lambda_1-\lambda_2)y]\tilde{P}_{\nu_1}(T=k,\ S_T\in dy)/g_{\nu_0,m}(\nu_0 m) \\
&\quad\le \exp[-(\lambda_1-\lambda_2)n\zeta]\sum_{k=1}^{[\frac{n}{2}]}\int_{n\zeta}^{n\zeta+1} g_{\nu_2,m-T}(1-y)\ \tilde{P}_{\nu_1}(T=k,\ S_T\in dy)/g_{\nu_0,m}(\nu_0 m).
\end{aligned}$$

From this step on the argument is the same as above. Substituting in the maximal value of $g_{\nu,m-T}$ and using Stirling's formula carefully, we arrive at

$$\tilde{P}^{(m)}_{\nu_0}\left(T<\frac{n}{2}\right)\le\sqrt{2}\ \exp\{-n[(\lambda_1-\lambda_2)\zeta-\phi(\lambda_2)]\}.$$

To complete the proof it is sufficient to show

Lemma 2.

$$\max_{\{(\theta_1,\theta_2):\ \psi(\theta_1)=\psi(\theta_2)\}}|(\theta_1-\theta_2)\zeta-\psi(\theta_2)|\ge 2\zeta^2$$

or equivalently

$$\max_{\{(\lambda_1,\lambda_2):\ \phi(\lambda_1)=\phi(\lambda_2)\}}|(\lambda_1-\lambda_2)\zeta-\phi(\lambda_2)|\ge 2\zeta^2.$$

Proof of Lemma 2. Using the method of Lagrange's multiplier, it is easy to show that $(\theta_1-\theta_2)\zeta-\psi(\theta_2)$ is maximised at θ_1 and θ_2 satisfying

$$\begin{cases} \frac{1}{\theta_1}+\frac{1}{|\theta_2|}=\frac{1}{\zeta} \\ \psi(\theta_1)=\psi(\theta_2) \end{cases}. \tag{4}$$

Equation (4) involves a transcendental equation which is difficult to solve explicitly, but here is an easy way out. Dvoretsky, Kiefer, and Wolfowitz (1956) proved

$$P\{D_n^-\ge\sqrt{n}\,\zeta\}\le C_1\,e^{-2n\zeta^2}.$$

Siegmund (1982) showed

$$P\{D_n^-\ge\sqrt{n}\,\zeta\}\sim C_2(\zeta)\,e^{-n|(\theta_1-\theta_2)-\psi(\theta_2)|}$$

where θ_1 and θ_2 satisfy(4) and $C_2(\varsigma)$ is a constant depending only on ς. These two results imply

$$\overline{\lim}_{n\to\infty} C_1\, e^{-2n\varsigma^2}/C_2(\varsigma)e^{-n|(\theta_1-\theta_2)\varsigma-\psi(\theta_2)|} \geq 1.$$

Suppose $(\theta_2-\theta_1)\varsigma-\psi(\theta_2) < 2\varsigma^2$ for some ς, then

$$\lim_{n\to\infty} C_1\, e^{-2n\varsigma^2}/C_2(\varsigma)\, e^{-n|(\theta_1-\theta_2)\varsigma-\psi(\theta_2)|} = 0.$$

This is a contradiction. Consequently Lemma 2 is true, and the proof of Theorem 1 is completed.

3. Concluding Remarks

(i) Theorem 1 is useful in determining confidence bounds, cf. Birnbaum and McCarty (1958) and Csörgő and Horváth (1981).

(ii) It is also possible to derive a bound of the form $P\{D_n^- \geq \sqrt{n}\,\varsigma\} \leq \varsigma^{-\frac{1}{2}}\, e^{-2n\varsigma^2}$ by working on (2) only. This bound is strictly better than Theorem 1 of $\varsigma > \frac{1}{8}$, but the result is poor when ς is small. This is the reason why we split the set $\{\tau < m\}$ into two parts and use a different argument on each part.

(iii) Birnbaum and Tingey (1951) gave the exact distribution of D_n^-, but their formula is inconvenient for numerical calculation. This is one of the reasons that a bound like Theorem 1 may be useful in application.

(iv) At first sight, the conjecture mentioned in section one seems unlikely to be true, when compared with the asymptotic result $\lim_{n\to\infty} P\{D_n^- > \varsigma\} = e^{-2\varsigma^2}$, but Smirnov's (1944) result $P\{D_n^- > \varsigma\} = \exp[-2\varsigma(\varsigma + (3n^{\frac{1}{2}})^{-1})] + o(n^{-\frac{1}{2}})$, which suggests that D_n^- approach the asymptotic distribution from below, served as analytical support of the conjecture.

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4. References

Birnbaum, Z. W. and McCarty, R. C. (1958). *A distribution-free upper confidence bound for $Pr\{Y < X\}$, based on independent samples of X and Y.* **Ann. Math. Statist.**, Vol. 29, 558-562.

Birnbaum, Z. W. and Tingey, F. H. (1951). *One-sided confidence contours for probability distribution function.* **Ann. Math. Statist.**, Vol. 22, 592-596.

Csörgő, S. and Horváth, L. (1981). *On the Koziol-Green model for random censorship.* **Biometrika**, Vol. 68, 391-401.

Devroye, L. P. and Wise, G. L. (1979). *On the recovery of discrete probability densities from imperfect measurements.* **J. Franklin Inst.**, Vol. 307, 1-20.

Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1956). *Asymptotic minimax character of the sample distribution function and of the multinomial estimator.* **Ann. Math. Statist.**, Vol. 27,642-669.

Feller, W. (1966). **An Introduction to Probability Theory and Its Applications I**, John Wiley & Sons, Inc., New York.

Siegmund,D. (1982). *Large deviations for boundary crossing probabilities.* **Ann. Prob.**, Vol. 10, 581-588.

Siegmund, D. (1983). *Corrected diffusion approximations and their applications.* Stanford University Technical Report.

Smirnov, N. V. (1944). *An approximation to the distribution laws of random quantities determined by empirical data.* **Uspehi Mat. Nauk**, Vol. 10, 179-206.